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A general problem of an optimal equivalent change of measure and contingent claim pricing in an incomplete market ☆

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Abstract

A general model of an optimal equivalent change of measure is considered. Existence and uniqueness conditions of a solution of backward semimartingale equation for the value process are given. This result is applied to determine the maximum price of a contingent claim.

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1. Introduction and the main results

We consider a general model of an optimal equivalent change of measure and derive Bellman's type equation for the value process. This equation contains Chitashvili's backward semimartingale equation for the value process in an optimal control problem (Chitashvili, 1983) and the El Karoui–Quenez approximating equations for the selling price of contingent claims (El Karoui and Quenez, 1995), which were derived in the case of Brownian filtration.

Let $(\Omega, \mathscr{F}, F = (F_t, t \in [0,T]), P)$ be a filtered probability space satisfying the usual conditions, where $T < \infty$ is a fixed time horizon. We assume that $\mathscr{F} = F_T$ and F_0 is P-trivial. Let $\mathscr{Q}$ be a family of probabilities on F_T equivalent to the measure P for all $Q \in \mathscr{Q}$.

For each $Q \in \mathscr{Q}$ we denote by $Z^Q = (Z_t^Q = \mathrm{d}Q_t/\mathrm{d}P_t,\ t \geqslant 0)$ the density process of the measure Q relative to P, where $Q_t = Q/F_t$, $P_t = P/F_t$ are restrictions of measures Q and P to the σ-algebra F_t. As is known, Z^Q is a uniformly integrable martingale with respect to the filtration $F = (F_t, t \in [0,T])$ and the measure P and there is a local martingale $M^Q \in \mathscr{M}_{\mathrm{loc}}(F,P)$ such that

$$Z^Q = \mathscr{E}(M^Q) = (\mathscr{E}_t(M^Q), t \geqslant 0), \quad Q \in \mathscr{Q},$$

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where $\mathscr{E}(M)$ denotes the unique solution of the linear stochastic equation

$$dX_t = X_{t-} dM_t, \quad X_0 = 1. \tag{1}$$

Since all further considerations are invariant relative to equivalent changes of measure we assume that $P \in \mathscr{Q}$.

Denote $M^{\mathscr{Q}} = (M^Q, Q \in \mathscr{Q}), Z^{\mathscr{Q}} = (Z^Q, Q \in \mathscr{Q})$.

Thus, the class $M^{\mathscr{Q}}$ contains the process $M^P = 0$ (by convention $P \in \mathscr{Q}$) and any element $M^Q \in M^{\mathscr{Q}}$ is characterized by the properties:

(1) $\Delta M_t^Q = M_t^Q - M_{t-}^Q > -1$, P-a.s., $t \in [0, T]$,

(2) $\mathscr{E}_t(M^Q)$, $t \in [0, T]$ is a P-martingale,

(3) any measure Q, equivalent to P, such that $dQ = \mathscr{E}_T(M)\, dP$ for some $M \in M^{\mathscr{Q}}$, belongs to the class $\mathscr{Q}$.

Assume that

(A) η is a F_T-measurable random variable such that

$$\sup_{Q \in \mathscr{Q}} E^Q|\eta| < \infty,$$

where E^Q stands for the mathematical expectation with respect to the measure Q.

The problem is to maximize the expected cost $V^Q = E^Q \eta$ by a suitable choice of an equivalent measure $Q \in \mathscr{Q}$. Consider the process

$$\operatorname*{esssup}_{Q \in \mathscr{Q}} E^Q(\eta / F_t), \quad t \in [0, T]. \tag{2}$$

Depending on the set of measures $\mathscr{Q}$, the process V can be understood as the value process of an optimal control problem (if $\mathscr{Q}$ is a set of controlled measures), or as the selling price of a contingent claim in an incomplete financial market model (if $\mathscr{Q}$ is a set of martingale measures for a discounted stock price process). The closeness of the class $\mathscr{Q}$ with respect to the bifurcation is a natural condition that can be imposed on the set of measures $\mathscr{Q}$ and which is satisfied for all natural classes of controlled measures (including, e.g., measures corresponding to a piecewise constant, usual or generalized controls, Chitashvili and Mania, 1987b) as well as for the set of martingale measures (see Lemma 4). This means that for any $Q_1, Q_2 \in \mathscr{Q}$, $t \in [0, T]$ and $B \in F_t$ there is a measure $Q \in \mathscr{Q}$ such that $Q(C) = P(C)$ for all $C \in F_t$,

$$Q(C/F_t) I_B = Q_1(C/F_t) I_B \quad \text{a.s. for any } C \in F_T,$$

and

$$Q(C/F_t) I_{B^c} = Q_2(C/F_t) I_{B^c} \quad \text{a.s. for any } C \in F_T.$$

For convenience, we formulate this condition in the following equivalent form in terms of martingales M^Q.

(B) For any $Q_1, Q_2 \in \mathscr{Q}$, $t \in [0, T]$ and F_t-measurable set B there exists $Q \in \mathscr{Q}$ such that $M_s^Q = 0$ if $s \leqslant t$ and

$$M_s^Q - M_t^Q = (M_s^{Q_1} - M_t^{Q_1}) I_B + (M_s^{Q_2} - M_t^{Q_2}) I_{B^c}, \quad s > t.$$

Condition (B) enables us to give a supermartingale characterization of the value process V. The following proposition is proved in a standard way (see e.g. El Karoui and Quenez (1995, Proposition A2) or Elliott (1982, Lemma 16.11)).

Proposition 1. *Let conditions* (A) *and* (B) *be satisfied. Then*

(a) *there exists a right continuous with left limits* (*RCLL*) *process* $V = (V_t, t \in [0,T])$ *such that for all* $t \in [0,T]$

$$V_t = \operatorname*{esssup}_{Q \in \mathscr{Q}} E^Q(\eta/F_t) \quad \text{a.s.} \tag{3}$$

The process V is the smallest right continuous supermartingale with respect to Q, for every $Q \in \mathscr{Q}$, which is equal to η at time T,

(b) *the measure Q^* is optimal* (*i.e.*, $V_t = E^{Q^*}(\eta/F_t)$ *for every* $t \in [0,T]$) *if and only if V is a martingale relative to the measure Q^*.*

The optimality principle in the form of Proposition 1 is a general statement which holds for processes with a sufficiently complex structure. It is difficult, however, to test optimality condition (b), and the natural desire to express this condition in predictable terms leads to the necessity to find a canonical decomposition of the value process relative to P (or relative to some other measure $Q \in \mathscr{Q}$) and to give a differential characterization of the value process, which is the task of this paper.

Let

$$V_t = V_0 + N_t - B_t, N \in \mathscr{M}_{\text{loc}}(P,F), B \in \mathscr{A}^+_{\text{loc}}(P,F) \cap \mathscr{P} \tag{4}$$

be the Doob–Meyer decomposition of the value process relative to the measure P.

Since V is a supermartingale with respect to any measure $Q \in \mathscr{Q}$, the square predictable characteristic $\langle M^Q, N \rangle$ always exists for any $Q \in \mathscr{Q}$ (Proposition 2) and Girsanov's theorem implies that

$$B_t - \langle M^Q, N \rangle_t \in \mathscr{A}^+_{\text{loc}} \quad \text{for any } Q \in \mathscr{Q}. \tag{5}$$

Our aim is to prove that B is the minimal increasing process with property (5) and to show that the value process V uniquely solves a backward semimartingale equation under additional assumptions (on the family of measures $\mathscr{Q}$) given below.

We say that the process B strongly dominates the process A and write $A \prec B$, if the difference $B - A$ is a locally integrable increasing process.

Denote by $((\operatorname{esssup}_{Q \in \mathscr{Q}} \langle M^Q, N \rangle)_t, t \in [0,T])$ the least increasing process, zero at time zero, which strongly dominates the process $(\langle M^Q, N \rangle_t, t \in [0,T])$ for every $Q \in \mathscr{Q}$, i.e., this is an 'ess sup' of the family $(\langle M^Q, N \rangle, Q \in \mathscr{Q})$ relative to the strong order $\prec$.

We shall use the following assumptions:

(C) A martingale part of any supermartingale Y with $Y_T = \eta$ a.s., is locally square integrable.

(D) $M^Q \in \mathscr{M}^2_{\text{loc}}$ for all $Q \in \mathscr{Q}$ and there exists a sequence $(\tau_n, n \geqslant 1)$ of stopping times with $\tau_n \uparrow T$ such that for any finite sequence $(Q_i, 1 \leqslant i \leqslant m) \in \mathscr{Q}$ and for any finite sequence $(\alpha^i, 1 \leqslant i \leqslant m)$ of positive predictable processes with $\sum_i^m \alpha_s(\omega) = 1$ (for any s and ω) we have for each $n \geqslant 1$

$$\left\langle \sum_i^m \int_0^\cdot \alpha_s^i \, \mathrm{d}M_s^{Q_i} \right\rangle_{\tau_n} \leqslant n. \tag{6}$$

(E) There exists a sequence $(s_n, n\geqslant 1)$ of stopping times with $s_n \uparrow T$ and a sequence $(c_n, n\geqslant 1)$ of real numbers with $c_n \downarrow -1$ such that a.s.

$$\operatorname*{essinf}_{Q\in\mathcal{Q}} \inf_{t\leqslant s_n} \Delta M_t^Q \geqslant c_n > -1.$$

(F) For any $m \in \mathcal{M}_{\mathrm{loc}}$ (for which $\langle M^Q, m\rangle$ exists for all $Q \in \mathcal{Q}$) there exists a predictable bounded increasing process L such that for any $\varepsilon > 0$ there is a measure $Q_\varepsilon \in \mathcal{Q}$ for which a.s.

$$\langle M^{Q_\varepsilon}, m\rangle_t + \varepsilon L_t - (\operatorname*{esssup}_{Q\in\mathcal{Q}}\langle M^Q, m\rangle)_t \in \mathcal{A}^+_{\mathrm{loc}}.$$

Remark 1. In particular, condition (D) implies that

$$\langle M^Q\rangle_{\tau_n} \leqslant n \quad \text{for all } Q \in \mathcal{Q}. \tag{7}$$

Condition (D) and (7) are equivalent if for any $(Q_i, 1\leqslant i\leqslant m)$ and $(\alpha^i, 1\leqslant i\leqslant m)$ from condition (D) there exists $Q \in \mathcal{Q}$ such that $M^Q = \sum_i^m \alpha^i \cdot M^{Q_i}$.

Note that condition (D) is satisfied if, e.g., the square characteristics $(\langle M^Q\rangle, Q \in \mathcal{Q})$ are strongly dominated by some locally integrable increasing process (see Remark 2 in Section 3).

Remark 2. Condition (F) is fulfilled if the class $\mathcal{Q}$ is closed with respect to the strong bifurcation (see Lemma 2).

For a special semimartingale X and a local martingale M we denote by $\langle X, M\rangle$ the predictable mutual characteristic of M and the martingale part of X.

Consider the backward semimartingale equation

$$\mathrm{d}Y_t = -\mathrm{d}\left(\operatorname*{esssup}_{Q\in\mathcal{Q}}\langle M^Q, Y\rangle\right)_t + \mathrm{d}m_t, \quad m \in \mathcal{M}_{\mathrm{loc}} \tag{8}$$

with the boundary condition

$$Y_T = \eta. \tag{9}$$

We say that the process Y is a solution of (8), (9) if Y is a P-supermartingale with the decomposition

$$Y_t = Y_0 + m_t - A_t,\ m \in \mathcal{M}_{\mathrm{loc}}(P, F),\ A \in \mathcal{A}^+_{\mathrm{loc}}(F)$$

such that

(i) $Y_T = \eta$ a.s.

(ii) $\langle m, M^Q\rangle$ exists for each $Q \in \mathcal{Q}$ and

$$A_t = \left(\operatorname*{esssup}_{Q\in\mathcal{Q}}\langle M^Q, m\rangle\right)_t, \quad t < T. \tag{10}$$

We recall that the process X is said to belong to class D if the random variables $X_\tau I_{(\tau\leqslant T)}$ for all stopping times τ are uniformly integrable. Denote by $D(\mathcal{Q})$ the class of processes which belong to the class D with respect to any measure $Q \in \mathcal{Q}$.

Now we formulate the main statement of this paper.

Theorem 1. (a) *Under conditions* (A)–(E) *the value process V is a solution of the equations* (8), (9).

(b) *If conditions* (A), (B) *and* (F) *are satisfied and if* (8), (9) *admit a solution in* $D(\mathcal{Q})$, *this solution is unique and is the value process.*

(c) *If conditions* (A), (B), (D)–(F) *are satisfied and one of the following conditions*

(A1) $E\eta^2 < \infty$ *and* $\langle M^Q\rangle_T \leqslant C$ *for all* $Q \in \mathcal{Q}$, *or*

(A2) $\sup_{Q\in\mathcal{Q}} E^Q\eta^2 < \infty$

is fulfilled, then the value function is the unique solution of (8), (9) *in the class* $D(\mathcal{Q})$.

Let us introduce some notions which enable us to apply this theorem to the optimal control problem.

Let $P^A = (P^a,\ a \in A)$ be a family of probability measures equivalent to the measure P on F_T, where A is a compact subset of some metric space. Denote by $M^A = (M^a, a \in A)$ the set of local martingales $(M_t^a,\ t \in [0,T])$ by means of which the elements of the set $R^A = (\rho^a, a \in A)$ of local densities $\rho^a = (\mathrm{d}P_t^a/\mathrm{d}P_t,\ t \in [0,T])$ are represented as exponential martingales $\rho = (\mathscr{E}_t(M^a),\ t \in [0,T])$.

The elements of the set A are interpreted as decisions and the class U of controls is defined as a set of predictable processes taking values in A. The problem of a definition of controlled measures in the case under consideration (i.e. for an arbitrary family of information flow and dominated family of probability measures) was solved by Chitashvili (1983) using the notion of a stochastic line integral. Following Chitashvili (1983), we define the controlled measure P^u, associated to any $u \in U$, by

$$\mathrm{d}P^u = \mathscr{E}_T(M^u)\,\mathrm{d}P, \tag{11}$$

where M^u is the stochastic line integral with respect to the family of martingales $(M^a,\ a \in A)$ (a definition of the stochastic line integral is given in Section 4. See Chitashvili and Mania (1987a) for details).

Let us consider the maximization problem $E^u\eta \to \max_u$ and let

$$S_t = \operatorname*{esssup}_{u\in U} E^u(\eta/F_t), \quad t \in [0,T]$$

be the value process, where E^u is the mathematical expectation with respect to the measure P^u. We assume (apart from the other conditions) that $M^a \in \mathscr{M}^2_{\mathrm{loc}}$ and $\langle M^a\rangle \ll K$, $a \in A$, for some predictable increasing process $K = (K_t,\ t \geqslant 0)$. Denote by $H(a,m) = \mathrm{d}\langle M^a, m\rangle/\mathrm{d}K$ the Hamiltonian of the problem.

We show (Theorem 3) that under conditions (C1)–(C6) (of Section 4) on the family $(M^a,\ a \in A)$ conditions (A)–(F) for the class of controlled measures $\mathcal{Q} = P^U$ are satisfied and that

$$\left(\operatorname*{esssup}_{u\in U}\langle M^u, V\rangle\right)_t = \int_0^t \sup_{a\in A} H(s,a,V)\,\mathrm{d}K_s \quad \text{a.s.} \tag{12}$$

Therefore, it follows from Theorem 1 that under conditions (C1)–(C6) the value process S solves the stochastic Bellman equation

$$\mathrm{d}Y_t = \left(\sup_{a\in A} H(t,a,y_t)\right)\mathrm{d}K_t + \mathrm{d}m_t, \quad m \in \mathscr{M}_{\mathrm{loc}} \tag{13}$$

with the boundary condition $Y_T = \eta$.

Note that this form of the backward semimartingale equation, which plays the role of Bellman's equation, was proposed by Chitashvili (1983). In Chitashvili (1983), using the successive approximation method, an existence of a solution of martingale equation, equivalent to (13), was proved. We prove an existence result for Eq. (8) (which contains (13)) without using any successive approximation procedures and do not impose the domination condition of square characteristics of martingales used in Chitashvili (1983) and Chitashvili and Mania (1987a,b). An explanation, why (13) is the semimartingale version of the Bellman equation one can see in Chitashvili and Mania (1996).

In Section 5 we apply Theorem 1 to derive the approximating equations for the maximum price of a contingent claim in a general incomplete market model. Assume that the market contains d securities whose discounted price process X is a vector valued RCLL locally bounded process. Denote by $\mathscr{P}(X)$ the set of local martingale measures and let η be the value of a contingent claim at maturity T.

Denote by V_t the maximum of the possible prices of the contingent claim η at time t

$$V_t = \operatorname*{esssup}_{Q \in \mathscr{P}(X)} E^Q(\eta/F_t), \quad t \in [0, T].$$

The class of a local martingale measures is stable with respect to bifurcation and, hence, V admits a supermartingale characterization by Proposition 1. Note that Theorem 1 is not directly applicable for the value process V, since condition (D) is not usually satisfied for the set $\mathscr{P}(X)$ of all martingale measures. But one can restrict the set $\mathscr{P}(X)$ to some subset $\mathscr{P}^n(X)$ of martingale measures, so that $V_t^n = \operatorname{esssup}_{Q \in \mathscr{P}^n(X)} E(\eta/F_t)$ tends to V_t and it is possible to determine V^n as a unique solution of a backward semimartingale equation (8), (9). This method was developed by El Karoui and Quenez (1995) in the context of Brownian model. Using Theorem 1 we generalize the corresponding result of El Karoui and Quenez (1995) for a market model with an arbitrary filtration and the locally bounded discounted asset price process.

All notations and the basic facts concerning the martingale theory used below can be found in Dellacherie and Meyer (1980), Jacod (1979) and Liptzer and Shiryayev (1986).

2. A differential characterization of the value process

For convenience we first prove the following known assertion.

Proposition 2. *Let conditions* (A) *and* (B) *are satisfied. Then* $\langle M^Q, N \rangle$ *exists for any* $Q \in \mathscr{Q}$ *and*

$$\langle M^Q, N \rangle \prec B, \quad Q \in \mathscr{Q},$$

where B *is a predictable increasing process and* N *is a martingale part in the Doob–Meyer decomposition* (4) *of the value process* V *relative to the measure* P.

Proof. Since B is predictable and V is a Q-supermartingale, the proceess $N=V-B$ will be a special semimartingale with respect to any $Q\in\mathscr{Q}$. Therefore, for any $Q\setminus$ in $\mathscr{Q}[M^Q,N]$ is locally P-integrable according to Jacod (1979) (Corollary 7.29).

Thus, $\langle M^Q,N\rangle$ exists for any $Q\in\mathscr{Q}$ and we can write

$$\operatorname*{esssup}_{Q\in\mathscr{Q}} E^Q(\eta/F_t)-\sup_{Q\in\mathscr{Q}} E^Q\eta=N_t-\langle M^Q,N\rangle_t-B_t+\langle M^Q,N\rangle_t.$$

Since V is a supermartingale relative to each $Q\in\mathscr{Q}$ and $N-\langle M^Q,N\rangle$ is a Q-local martingale by Girsanov's theorem, $B_t-\langle M^Q,N\rangle_t$ will be an increasing process for any $Q\in\mathscr{Q}$. □

Proof of Theorem 1. (a) By Proposition 2 we have that

$$B_t-\left(\operatorname*{esssup}_{Q\in\mathscr{Q}}\langle M^Q,N\rangle\right)_t\in\mathscr{A}^+_{\mathrm{loc}}. \tag{14}$$

Let us show now that

$$B_t=\left(\operatorname*{esssup}_{Q\in\mathscr{Q}}\langle M^Q,N\rangle\right)_t,\quad t<T,\ \text{a.s.} \tag{15}$$

For any $\varepsilon>0$ there exists $Q_\varepsilon\in\mathscr{Q}$ such that

$$E^{Q_\varepsilon}\eta+\varepsilon>V_0=\sup_{Q\in\mathscr{Q}}E^Q\eta.$$

Therefore, for each $t\in[0,T]$

$$E^{Q_\varepsilon}(\eta/F_t)-E^{Q_\varepsilon}\eta-\varepsilon\leqslant V_t-V_0=N_t-B_t$$

and

$$E^{Q_\varepsilon}(\eta/F_t)-E^{Q_\varepsilon}\eta-(N_t-\langle M^{Q_\varepsilon},N\rangle_t)-\varepsilon\leqslant\langle M^{Q_\varepsilon},N\rangle_t-B_t. \tag{16}$$

It follows from conditions (C)–(E) and from the locally boundedness of B that there exists a sequence of stopping times $(\sigma_n,n\geqslant 1)$ with $\sigma_n\uparrow T$ (P-a.s.) such that for any $(Q_i,1\leqslant i\leqslant m)$ and $(\alpha^i,1\leqslant i\leqslant m)$ from condition (D)

$$\left\langle\sum_i^m\int_0^\cdot\alpha^i_s\,\mathrm{d}M^{Q_i}_s\right\rangle_{\sigma_n}\leqslant n,\quad \operatorname*{essinf}_{Q\in\mathscr{Q}}\inf_{t\leqslant\sigma_n}\Delta M^Q_t\geqslant c_n>-1,\quad B_{\sigma_n}\leqslant n,\ \langle N\rangle_{\sigma_n}\leqslant n. \tag{17}$$

For any $Q\in\mathscr{Q}$ the process $N_{t\wedge\sigma_n}-\langle M^Q,N\rangle_{t\wedge\sigma_n}$ is a uniformly integrable martingale with respect to the measure Q. Indeed, since by (17) $\langle M^Q\rangle_{\sigma_n}\leqslant n$ for all $Q\in\mathscr{Q}$ we have that (see Jacod, 1979, Proposition 8.27)

$$E\mathscr{E}^2_{\sigma_n}(M^Q)\leqslant e^{2n}\quad\text{for all } Q\in\mathscr{Q}, \tag{18}$$

and using successively (18), the Hölder, Doob and Kunita–Watanabe inequalities, we obtain that

$$\begin{aligned}E^Q\sup_{t\leqslant\sigma_n}|N_t-\langle M^Q,N\rangle_t|&\leqslant E^{1/2}\mathscr{E}^2_{\sigma_n}(M^Q)E^{1/2}\sup_{t\leqslant\sigma_n}(N_t-\langle M^Q,N\rangle_t)^2\\&\leqslant 2e^{2n}\left(E^{1/2}\sup_{t\leqslant\sigma_n}N_t^2+E^{1/2}\langle M^Q\rangle_{\sigma_n}\langle N\rangle_{\sigma_n}\right)\leqslant C(n)<\infty.\end{aligned} \tag{19}$$

Thus, $N_{t\wedge\sigma_n} - \langle M^{Q_\varepsilon}, N\rangle_{t\wedge\sigma_n}$ is a uniformly integrable martingale relative to the measure Q_ε and the localizing sequence σ_n does not depend on ε. Since $E^{Q_\varepsilon}(\eta/F_t)$ is also a martingale, taking expectations (with respect to the measure Q_ε) in (16) for the stopped processes we have

$$E^{Q_\varepsilon}(\langle M^{Q_\varepsilon}, N\rangle_{\sigma_n\wedge t} - B_{\sigma_n\wedge t}) \geqslant -\varepsilon$$

and, hence,

$$E\mathscr{E}_{\sigma_n\wedge t}(M^{Q_\varepsilon})\left(\left(\operatorname*{esssup}_{Q\in Q}\langle M^Q, N\rangle\right)_{\sigma_n\wedge t} - B_{\sigma_n\wedge t}\right) \geqslant -\varepsilon. \tag{20}$$

Let $(\varepsilon_i,\ i\geqslant 1)$ be a sequence of positive numbers converging to zero. Since $E\mathscr{E}^2_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}})\leqslant e^{2n}$ for all $i\geqslant 1$, the sequence $(\mathscr{E}_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}}),\ i\geqslant 1)$ is weakly compact subset of $L^2(F_{\sigma_n\wedge t})$ for any $t\in[0,T]$ and $n\geqslant 1$. Therefore, there is a subsequence of $(\mathscr{E}_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}}),\ i\geqslant 1)$ (for convenience we preserve the same notations for the selected subsequence) weakly converging to some $\rho\in L^2(F_{\sigma_n\wedge t})$. Passing to the limit in (20) as $i\to\infty$ we obtain that

$$E\rho\left(\left(\operatorname*{esssup}_{Q\in\mathscr{Q}}\langle M^Q, N\rangle\right)_{\sigma_n\wedge t} - B_{\sigma_n\wedge t}\right) \geqslant 0. \tag{21}$$

Let us show that $\rho>0$ a.s. It is obvious that $E\rho=1$ and let $\tilde{P}$ be a measure absolutely continuous relative to P defined by $\mathrm{d}\tilde{P}=\rho\,\mathrm{d}P$. The density process $\rho_t=\mathrm{d}\tilde{P}_t/\mathrm{d}P_t$ (evidently, $\rho_T=\rho_{\sigma_n}=\rho$, since ρ is F_{σ_n}-measurable) is representable as an exponential martingale $\mathscr{E}(M)$. Note that the present assumptions do not imply that $M\in M^{\mathscr{Q}}$ (and, hence $\tilde{P}\in\mathscr{Q}$), but we need only to show that the measure $\tilde{P}$ is equivalent to P on F_{σ_n}, which implies that $\rho>0$ P-a.s. Since $(\mathscr{E}_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}}),\ i\geqslant 1)$ weakly converges to $\mathscr{E}_{t\wedge\sigma_n}(M)$, there exist convex combinations of $(\mathscr{E}_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}}),\ i\geqslant 1)$ which converge strongly to the same limit. Applying Lemma A.1 and condition (D) we have that each convex combination of $(\mathscr{E}_{\sigma_n\wedge t}(M^{Q_{\varepsilon_i}}), i\leqslant j)$ is represented as $\mathscr{E}(M^j)$ with $\langle M^j\rangle_{\sigma_n}\leqslant n$. Therefore, for any fixed $n\geqslant 1$ we have the convergence

$$E(\mathscr{E}_{\sigma_n}(M^j)-\mathscr{E}_{\sigma_n}(M))^2\to 0,\quad j\to\infty$$

and it follows from Proposition A1 that for any $\varepsilon>0$

$$P(\langle M^j - M\rangle_{\sigma_n}\geqslant\varepsilon)\to 0\quad\text{as } j\to\infty.$$

But this implies that $\langle M\rangle_{\sigma_n}\leqslant n$ and $\inf_{t\in[0,\sigma_n]}\Delta M_t>-1$. Since by Lenglart's inequality $\sup_{t\leqslant\sigma_n}|M^j_t-M_t|\to 0$ in probability (and a.s. for some subsequence), the latter relation follows from condition (E) and construction of M^j (see Lemma A.1), which imply that $\inf_j\inf_{t\leqslant\sigma_n}\Delta M^j_t\geqslant c_n>-1$. Hence, the measures $\tilde{P}$ and P are equivalent on F_{σ_n} for each $n\geqslant 1$ and $\rho>0$ P-a.s.

Let us return now to the proof of equality (15). Since $B_t-(\operatorname{esssup}_{Q\in\mathscr{Q}}\langle M^Q, N\rangle)_t\geqslant 0$ and $\rho>0$ P-a.s. we obtain from (21) that a.s.

$$B_{t\wedge\sigma_n}=\left(\operatorname*{esssup}_{Q\in\mathscr{Q}}\langle M^Q, N\rangle\right)_{\sigma_n\wedge t}.$$

Since $\sigma_n\uparrow T$, the passage to the limit results equality (15) for any $t<T$.

Proof of assertion (b): Let Y be a solution of Eqs. (8), (9) from the class $D(\mathscr{Q})$. Then Y is a P-supermartingale with the decomposition

$$Y_t - Y_0 = m_t - A_t, \quad m \in \mathscr{M}, \ A \in \mathscr{A}^+ \cap \mathscr{P}$$

and $\langle M^Q, m\rangle$ exists for all $Q \in \mathscr{Q}$. We have

$$Y_t = m_t - \langle m, M^Q\rangle_t - A_t + \langle m, M^Q\rangle_t,$$

where $m_t - \langle m, M^Q\rangle_t$ is a Q-local martingale by Girsanov's theorem and (10) implies that $A_t - \langle m, M^Q\rangle_t$ is an increasing process for every $Q \in \mathscr{Q}$. Therefore, Y will be a supermartingale of class D with respect to each $Q \in \mathscr{Q}$ and using the boundary condition (9) we obtain

$$Y_t \geqslant E^Q(Y_T/F_t) = E^Q(\eta/F_t)$$

for every $Q \in \mathscr{Q}$, hence,

$$Y_t \geqslant \operatorname*{esssup}_{Q \in \mathscr{Q}} E^Q(\eta/F_t), \quad t \in [0,T]. \tag{22}$$

Let us show the inverse inequality. By condition (F) for any $\varepsilon > 0$ there exists $Q_\varepsilon \in \mathscr{Q}$ such that

$$C_t = \langle M^{Q_\varepsilon}, m\rangle_t + \varepsilon L_t - \left(\operatorname*{esssup}_{Q \in \mathscr{Q}} \langle M^Q, m\rangle\right)_t \in \mathscr{A}^+_{\mathrm{loc}}. \tag{23}$$

Therefore, since Y solves (8),

$$Y_t = C_t - \langle M^{Q_\varepsilon}, m\rangle_t - \varepsilon L_t + m_t, \quad t < T$$

and Girsanov's theorem implies that the process $N_t = Y_t - C_t + \varepsilon L_t$ is a local martingale relative to the measure Q^ε, besides $\varepsilon L_t - C_t$ is a predictable increasing process. Thus, Y is a Q_ε supermartingale of class $D(Q_\varepsilon)$ and by uniqueness of the Doob–Meyer decomposition N will be a uniformly integrable martingale. Therefore, using the boundary condition (9) we have that

$$Y_t - C_t + \varepsilon L_t = E^{Q_\varepsilon}(\eta - C_T + \varepsilon L_T/F_t).$$

Since C_t is an increasing process, the last equality implies that

$$Y_t \leqslant E^{Q_\varepsilon}(\eta + \varepsilon(L_T - L_t)/F_t) \leqslant \operatorname*{esssup}_{Q \in \mathscr{Q}} E^Q(\eta + \varepsilon(L_T - L_t)/F_t)$$

and by arbitrariness of $\varepsilon > 0$ we obtain the inverse inequality

$$Y_t \leqslant \operatorname*{esssup}_{Q \in \mathscr{Q}} E^Q(\eta/F_t), \quad t \in [0,T],$$

hence $Y = V$.

Proof of assertion (c): Let (A1) be satisfied. Similarly to Proposition 8.27 of Jacod (1979) one can show that

$$E((\mathscr{E}_T(M^Q)/\mathscr{E}_\tau(M^Q))^2/F_\tau) \leqslant e^{2c} \quad \text{a.s.}$$

for any stopping time τ and any $Q \in \mathscr{Q}$. Using this inequality and the Hölder inequality we obtain that for any $Q \in \mathscr{Q}$

$$E^Q|V_\tau| I_{(|V_\tau| \geqslant \lambda)} \leqslant \text{const.}\, E^{1/2} \eta^2 I_{(|V_\tau| \geqslant \lambda)} \tag{24}$$

and

$$\sup_{t\in[0,T]} EV_t^2 \leqslant \text{const.}\, E\eta^2. \tag{25}$$

On the other hand, since $\operatorname{esssup}_{\tilde{Q}\in\mathcal{Q}} E^{\tilde{Q}}(|\eta|/F_t)$ is a Q-supermartingale, condition (A) and the Chebyshev inequality imply that

$$Q(|V_\tau|\geqslant\lambda)\leqslant\frac{1}{\lambda}E^Q \operatorname*{esssup}_{\tilde{Q}\in\mathcal{Q}} E^{\tilde{Q}}(|\eta|/F_t)\leqslant\frac{\sup_{Q\in\mathcal{Q}}E^Q|\eta|}{\lambda}\to 0,\quad \lambda\to\infty$$

for any $Q\in\mathcal{Q}$, uniformly relative to τ. Therefore, it follows from (24) that $V\in D(Q)$ for every $Q\in\mathcal{Q}$. The fact that $N\in\mathcal{M}^2_{\text{loc}}$ follows from decomposition (4), inequality (25) and from the locally boundedness of B.

The proof of (A2) $\Rightarrow V\in D(\mathcal{Q})$, $N\in\mathcal{M}^2_{\text{loc}}$ is evident.

Note that condition (C) was used in the proof of assertion (a) only for $N\in\mathcal{M}^2_{\text{loc}}$. Therefore, the proof of assertion (c) follows from assertions (a) and (b) of this theorem. □

Remark. It should be mentioned that, although the martingale m entering (8) looks like an unknown supplementary to Y, it is, however, uniquely determined by the boundary condition.

Corollary 1. *Let conditions* (A)–(F) *be satisfied. Then the measure Q^* is optimal if and only if V is of class D with respect to the measure Q^* and*

$$\langle M^{Q^*}, V\rangle_t = \left(\operatorname*{esssup}_{Q\in\mathcal{Q}}\langle M^Q, V\rangle\right)_t \quad \text{a.s. } t<T. \tag{26}$$

If in addition condition (A1) (*or* (A2)) *is satisfied then Q^* is optimal iff* (26) *holds.*

Proof. Theorem 1 and Girsanov's theorem imply that V is a local martingale under the measure Q^* if and only if (26) is satisfied. Therefore, the corollary follows from the fact that any local martingale belongs to the class D iff it is a uniformly integrable martingale. Note that $\langle M^Q, V\rangle = \langle M^Q, N\rangle$ for any Q, since N is a martingale part of V under P.

3. A backward semimartingale equation in strongly dominated case

Thus, Theorem 1 gives a characterization of an increasing process associated with the value process V and this enables to represent V as a solution of backward semimartingale equations (8), (9). In the strongly dominated case, considered in this section, it is possible to give this equation in a more explicit form.

Let M^Q is a locally square integrable martingale for any $Q\in\mathcal{Q}$ and there exists a predictable locally integrable increasing process $K=(K_t, t\geqslant 0)$ which dominates the square characteristics of the martingales M^Q, $Q\in\mathcal{Q}$.

For any $m\in\mathscr{M}_{\text{loc}}(P,F)$, for which the mutual characteristic $\langle M^Q,m\rangle$ exists, denote by $H(Q,m)$ and $H(Q)$, the Radon–Nicodým derivatives

$$\mathrm{d}\langle M^Q,m\rangle/\mathrm{d}K \quad \text{and} \quad \mathrm{d}\langle M^Q,M^Q\rangle/\mathrm{d}K,$$

respectively.

Let μ^K be the Doleans measure of the process K.

In this section we assume that instead of (D) and (F) the following (stronger) conditions (D*) and (F*) are satisfied:

(D*) $M^Q\in\mathscr{M}^2_{\text{loc}}(F,P)$, for some $K\in\mathscr{A}^+_{\text{loc}}\langle M^Q,M^Q\rangle\ll K$ for all $Q\in\mathscr{Q}$ and the process

$$\tilde K_t=\int_0^t \operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q)\,\mathrm{d}K_s,\quad t\in[0,T]$$

is locally integrable. Here we take the 'esssup' relative to the measure μ^K.

(F*) For any $m\in\mathscr{M}_{\text{loc}}$ (for which $\langle M^Q,m\rangle$ exists for all $Q\in\mathscr{Q}$) and any $\varepsilon>0$ there is a measure $Q_\varepsilon\in\mathscr{Q}$ such that μ^K-a.e.

$$H(Q_\varepsilon,m)>\operatorname*{esssup}_{Q\in\mathscr{Q}} H(Q,m)-\varepsilon. \tag{27}$$

Remark 1. It is evident that the process $\tilde K$ does not depend on the choice of the dominating process K and that $\tilde K-\langle M^Q\rangle\in\mathscr{A}^+_{\text{loc}}$ for all $Q\in\mathscr{Q}$. So, in fact, $\tilde K$ strongly dominates the family $\langle M^Q\rangle$, $Q\in\mathscr{Q}$.

Remark 2. Since for any $Q,Q'\in\mathscr{Q}$

$$\frac{\mathrm{d}\langle M^Q,M^{Q'}\rangle}{\mathrm{d}K}\leqslant\left(\frac{\mathrm{d}\langle M^Q\rangle}{\mathrm{d}K}\right)^{1/2}\left(\frac{\mathrm{d}\langle M^{Q'}\rangle}{\mathrm{d}K}\right)^{1/2}\leqslant\operatorname*{esssup}_{Q\in\mathscr{Q}} H(Q)\quad \mu^K\text{-a.e.}$$

we have that

$$\begin{aligned}&\int_0^t \operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q)\,\mathrm{d}K_s-\left\langle\sum_i^m\int_0^\cdot \alpha_s^i\,\mathrm{d}M_s^{Q_i}\right\rangle_t\\&\quad=\sum_{i=1}^m\sum_{j=1}^m\int_0^t\alpha_s^i\alpha_s^j\left(\operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q)-\frac{\mathrm{d}\langle M^{Q_i},M^{Q_j}\rangle_s}{\mathrm{d}K_s}\right)\mathrm{d}K_s\in\mathscr{A}^+_{\text{loc}}\end{aligned}\tag{28}$$

for any finite sequence $(Q_i,1\leqslant i\leqslant m)\in\mathscr{Q}$ and for any finite sequence $(\alpha^i,1\leqslant i\leqslant m)$ of positive predictable processes with $\sum_i^m\alpha_s(\omega)=1$. So, it is evident that (D*) implies (D).

Lemma 1. *Let condition* (D*) *be satisfied. For any local martingale m for which $\langle M^Q,m\rangle$ exists for every $Q\in\mathscr{Q}$ the process*

$$\int_0^t \operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q,m)\,\mathrm{d}K_s,\quad t\in[0,T],$$

is locally integrable. The essential supremum is understood here with respect to the measure μ^K.

Proof. Let $\mathscr{L}^2(M^{\mathscr{Q}})$ be a stable space of martingales generated by the family $(M^Q, Q\in\mathscr{Q})$ (see Jacod (1979) for definition and related results). Any $m\in\mathscr{M}_{\mathrm{loc}}$, for which $\langle M^Q,m\rangle$ exists for all $Q\in\mathscr{Q}$, may be expanded as a sum

$$m_t=\tilde{m}_t+m'_t,\quad t\in[0,T],$$

where $\tilde{m}\in\mathscr{L}^2_{\mathrm{loc}}(M^{\mathscr{Q}})$ and $\langle m',N\rangle=0$ for any $N\in\mathscr{L}^2_{\mathrm{loc}}(M^{\mathscr{Q}})$. It is easy to show that $\langle\tilde{m}\rangle\ll K$. Therefore, using the Kunita–Watanabe inequality for the Radon–Nicodym densities and the Höllder inequality, we have for any stopping time τ that

$$\int_0^\tau \operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q,m)\,\mathrm{d}K_s\leqslant\langle\tilde{m}\rangle_\tau^{1/2}\left(\int_0^\tau \operatorname*{esssup}_{Q\in\mathscr{Q}} H(s,Q)\,\mathrm{d}K_s\right)^{1/2}, \tag{29}$$

which implies the assertion of lemma. $\square$

Let us show that (F*) implies (F) and that condition (F*) is satisfied if the class $\mathscr{Q}$ is closed with respect to the strong bifurcation.

We say that the class $\mathscr{Q}$ is closed with respect to the strong bifurcation if the following condition is satisfied:

(B*) for any sequence $(Q_i,\ i\geqslant1)\in\mathscr{Q}$ and any sequence of predictable sets $(B_i,i\geqslant1)$, with $B_i\cap B_j=\emptyset$ for $i\neq j$ and $\bigcup_i B_i=[0,T]\times\Omega$, there exists $Q\in\mathscr{Q}$ such that

$$\int_0^T I_{B_i}(s)\,\mathrm{d}M_s^Q=\int_0^T I_{B_i}(s)\,\mathrm{d}M_s^{Q_i} \tag{30}$$

for every $i\geqslant1$.

Lemma 2. (D*), (B*) $\Rightarrow$ (F*) $\Rightarrow$ (F).

Proof. Let us first show the implication (F*) $\Rightarrow$ (F).

Let $\hat{K}_t=\int_0^t[1/(1+K_s)^2]\,\mathrm{d}K_s$, m is a local martingale such that $\langle M^Q,m\rangle$ exists for all $Q\in\mathscr{Q}$ and let $\hat{m}=\int_0^t(1+K_s)^2\,\mathrm{d}m_s$. Since K is locally bounded, $\hat{m}\in\mathscr{M}_{\mathrm{loc}}$ and $\langle M^Q,\hat{m}\rangle$ exists for all $Q\in\mathscr{Q}$. It is easy to see that μ^K-a.e.

$$\frac{\mathrm{d}\langle M^Q,\hat{m}\rangle}{\mathrm{d}K}=\frac{\mathrm{d}\langle M^Q,m\rangle}{\mathrm{d}\hat{K}}$$

on the set $[0,\tau_n]$, where $(\tau_n,n\geqslant1)$ is a sequence of stopping times such that $\tau_n\uparrow T$ and $K_{\tau_n}\leqslant n$ for every $n\geqslant1$.

Therefore, it follows from condition (F*) that

$$\frac{\mathrm{d}\langle M^{Q_\varepsilon},m\rangle}{\mathrm{d}\hat{K}}+\varepsilon>\operatorname*{esssup}_{Q\in\mathscr{Q}}\frac{\mathrm{d}\langle M^Q,m\rangle}{\mathrm{d}\hat{K}}$$

μ^K-a.e. on the set $[0,\tau_n]$ for all $n\geqslant1$ and this implies that

$$\langle M^{Q_\varepsilon},m\rangle_t+\varepsilon\hat{K}_t-\int_0^t\operatorname*{esssup}_{Q\in\mathscr{Q}}\frac{\mathrm{d}\langle M^Q,m\rangle_s}{\mathrm{d}\hat{K}_s}\,\mathrm{d}\hat{K}_s\in\mathscr{A}^+_{\mathrm{loc}}.$$

Since $\hat{K}$ is bounded and $\int_0^t\operatorname{esssup}_{Q\in\mathscr{Q}}(\mathrm{d}\langle M^Q,m\rangle_s/\mathrm{d}\hat{K}_s)\,\mathrm{d}\hat{K}_s-(\operatorname{esssup}_{Q\in\mathscr{Q}}\langle M^Q,m\rangle)_t\in\mathscr{A}^+_{\mathrm{loc}}$, we obtain that condition (F) is satisfied.

Now, let us show implication (D*), (B*) $\Rightarrow$ (F*). For a fixed $\varepsilon > 0$ let

$$B_Q = \left\{ (\omega, t) : H(t, Q, m) > \operatorname*{esssup}_{Q \in \mathscr{Q}} H(t, Q, m) - \varepsilon \right\}. \tag{31}$$

Let us introduce a partial order on the set $\mathscr{X} = \{(Q, B_Q) \colon Q \in \mathscr{Q}\}$ in the following way: $(q, B_q) < (Q, B_Q)$ if and only if

$$(1)\ B_q \subset B_Q, \quad (2) \int_0^T I_{B_q}(s)\,\mathrm{d}M_s^q = \int_0^T I_{B_q}(s)\,\mathrm{d}M_s^Q, \quad (3)\ \mu^K(B_p) < \mu^K(B_Q). \tag{32}$$

Now the proof is similar to that of Lemma 3.1 from Davis and Varaiya (1973) if we take into account the closeness of the class $\mathscr{Q}$ with respect to the strong bifurcation and Lemma 1, i.e., using the Zorn lemma one can show an existence of a maximal element (Q^*, B_{Q^*}) which will have the property $\mu^K(B^c_{Q^*}) = 0$, implying the assertion of lemma.

Lemma 3. *Let conditions* (B), (D*) *and* (F*) *be satisfied. Then for any local martingale m for which $\langle M^Q, m\rangle$ exists for all $Q \in \mathscr{Q}$*

$$\left(\operatorname*{esssup}_{Q \in \mathscr{Q}} \langle M^Q, m\rangle \right)_t = \int_0^t \operatorname*{esssup}_{Q \in \mathscr{Q}} \frac{\mathrm{d}\langle m, M^Q\rangle_s}{\mathrm{d}K_s}\,\mathrm{d}K_s \quad a.s.$$

Proof. Note that $A_t = \int_0^t \operatorname{esssup}_{Q \in \mathscr{Q}} (\mathrm{d}\langle m, M^Q\rangle_s / \mathrm{d}K_s)\,\mathrm{d}K_s$ is an increasing process by convention $P \in \mathscr{Q}$, since this implies that the class $M^{\mathscr{Q}}$ contains the process $X = 0$ (otherwise A will be a process of finite variation). It follows from Lemma 1 that A is locally integrable and it is obvious that $A_t - \langle M^Q, m\rangle_t \in \mathscr{A}^+_{\mathrm{loc}}$ for every $Q \in \mathscr{Q}$.

We should show that if for some $B \in \mathscr{A}^+_{\mathrm{loc}}$ the difference $B_t - \langle M^Q, m\rangle_t \in \mathscr{A}^+_{\mathrm{loc}}$ for every $Q \in \mathscr{Q}$ then $B_t - A_t \in \mathscr{A}^+_{\mathrm{loc}}$.

Suppose that $B_t - A_t \notin \mathscr{A}^+_{\mathrm{loc}}$. Then there exist a pair $s < t$ and $\delta > 0$ such that

$$P\left\{ B_t - B_s - \int_s^t \operatorname*{esssup}_{Q \in \mathscr{Q}} \frac{\mathrm{d}\langle m, M^Q\rangle_u}{\mathrm{d}K_u}\,\mathrm{d}K_u \leqslant -\delta \right\} > 0. \tag{33}$$

Condition (F*) implies that for every $\varepsilon > 0$ there exists $Q_\varepsilon \in \mathscr{Q}$ for which

$$\int_s^t \operatorname*{esssup}_{Q \in \mathscr{Q}} \frac{\mathrm{d}\langle m, M^Q\rangle_u}{\mathrm{d}K_u}\,\mathrm{d}K_u \leqslant \langle M^{Q_\varepsilon}, m\rangle_t - \langle M^{Q_\varepsilon}, m\rangle_s + \varepsilon(K_t - K_s). \tag{34}$$

Therefore, (33) implies that

$$P\{B_t - B_s - (\langle M^{Q_\varepsilon}, m\rangle_t - \langle M^{Q_\varepsilon}, m\rangle_s) \leqslant \varepsilon(K_t - K_s) - \delta\} > 0. \tag{35}$$

Since K is locally bounded, by arbitrariness of ε, we obtain that the process $B - \langle M^{Q_\varepsilon}, m\rangle$ is not increasing for sufficiently small ε and this contradicts the assumption that B strongly dominates $\langle M^Q, m\rangle$ for every $Q \in \mathscr{Q}$. $\square$

Consider now the backward semimartingale equation

$$Y_t = Y_0 - \int_0^t \operatorname*{esssup}_{Q \in \mathscr{Q}} \frac{\mathrm{d}\langle M^Q, Y\rangle_s}{\mathrm{d}K_s}\,\mathrm{d}K_s + m_t, \quad t < T \tag{36}$$

with the boundary condition $Y_T = \eta$.

Theorem 2. (a) *Under conditions* (A)–(C), (D*), (E) *and* (F*) *the value process* V *is a solution of Eq.* (36).

(b) *If conditions* (A), (B), (D*), (F*) *are satisfied then the solution of* (36) *is unique in the class* $D(\mathscr{Q})$, *if it exists.*

(c) *If* (A), (B), (D*), (E), (F*) *and one of conditions* (A1) *or* (A2) *are satisfied, then the value process* V *is the unique solution of* (36) *in the class* $D(\mathscr{Q})$.

Besides, the measure Q^* *is optimal if and only if* μ^K*-a.e.*

$$H(Q^*, V) = \operatorname*{esssup}_{Q \in \mathscr{Q}} H(Q, V). \tag{37}$$

Proof. Since (D*) ⇒ (D) (see Remark 2 of this section) and (F*) ⇒ (F) (Lemma 2), the proof of this theorem follows from Theorem 1 and Lemma 3. □

4. Application to optimal control. A derivation of the Bellman–Chitashvili equation

Let $P^A = (P^a, a \in A)$ be a family of probability measures on F_T equivalent to the measure P. Assume that P_0^a is the same for all $a \in A$ and, without any loss of generality, let $P_0^a = P_0$.

Denote by $R^A = (\rho^a, a \in A)$ the set of local densities $\rho^a = (\mathrm{d}P_t^a/\mathrm{d}P_t,\ t \in [0,T])$ and let $M^A = (M^a, a \in A)$ be a set of local martingales such that $\rho^a = (\mathscr{E}_t(M^a),\ t \in [0,T]),\ a \in A$.

Suppose that the decision set A is a compact subset of a metric space. The possibility of decision change with respect to the accumulated information leads to the extension of the class P^A by introducing controls. The class U of controls is defined as a set of predictable processes taking values in A and the set of probabilities corresponding to controls is generated by the operations of bifurcation and closure.

Suppose that the following conditions are satisfied:

(C2) $M^a \in \mathscr{M}^2_{\mathrm{loc}}(P, F)$ and $\langle M^a \rangle \ll K$ for all $a \in A$ for some $K \in \mathscr{A}^+_{\mathrm{loc}}$.

For any $a \in A$ and for any $m \in \mathscr{M}_{\mathrm{loc}}$ for which $\langle M^a, m \rangle$ exists denote by $H(a, m)$ the Radon–Nicodým derivative

$$\mathrm{d}\langle M^a, m \rangle / \mathrm{d}K$$

and for any $a, b \in A$ let

$$\varphi(a,b) = \mathrm{d}\langle M^a, M^b \rangle / \mathrm{d}K, \quad \varphi(a) = \mathrm{d}\langle M^a, M^a \rangle / \mathrm{d}K. \tag{38}$$

(C3) $\int_0^t \sup_{a \in A} \varphi(s,a)\, \mathrm{d}K_s \in \mathscr{A}^+_{\mathrm{loc}}$.

Sometimes we shall use the more strong condition

(C3*) $\int_0^T \sup_{a \in A} \varphi(s,a)\, \mathrm{d}K_s \leqslant C < \infty$, a.s.

(C4) The Radon–Nicodým derivative $\varphi(a,b) = \mathrm{d}\langle M^a, M^b \rangle / \mathrm{d}K$ is a continuous function of (a,b) for almost every couple (ω, t) with respect to the measure μ^K, where μ^K is the Doleans measure of K.

(C5) ΔM_t^a is continuous in a a.s. uniformly with respect to t.

(C6) $\inf_{a \in A} \inf_{t \in [0,T]} \Delta M_t^a \geqslant c > -1$, a.s.

It follows from Chitashvili and Mania (1987a) (Lemma 1) that condition (C4) implies the existence of μ^K-a.e. continuous in a modification of the function $H(a,m)$ and such a version will be considered.

Denote by $\psi(a,b)$ the Radon–Nicodým derivative $\mathrm{d}\langle M^a - M^b\rangle/\mathrm{d}K$. Note that

$$H(a,M^b) = \varphi(a,b) \quad \text{and} \quad \psi(a,b) = \varphi(a) - 2\varphi(a,b) + \varphi(b), \quad \mu^K\text{-a.e.} \tag{39}$$

We give now the definition of a controlled process, which is based on the notion of a stochastic line integral, suggested by Chitashvili (1983).

The measure P^u corresponding to any control $u \in U$ is constructed by the following chain of transitions:

$$\begin{aligned} P^A \to R^A = (\rho^a = \mathrm{d}P^a/\mathrm{d}P, a \in A) \to M^A = (M^a, a \in A) \to M^U = (M^u, u \in U) \\ \to R^U = (\rho^u = \mathscr{E}(M^u), u \in U) \to P^U = (P^u = \rho^u \cdot P, u \in U), \end{aligned} \tag{40}$$

where the stochastic line integral M^u is the determining step.

Let $L_{\mathrm{loc}}(U) = \{u \in U\colon \int_0^t \varphi(s,u_s)\,\mathrm{d}K_s \in \mathscr{A}^+_{\mathrm{loc}}\}$.

Under conditions (C2), (C4) the stochastic line integral M^u is defined (for any $u \in L_{\mathrm{loc}}(U)$) as a unique element of the stable space of martingales $\mathscr{L}^2_{\mathrm{loc}}(M^a, a \in A)$ such that for every $m \in \mathscr{M}^2_{\mathrm{loc}}$

$$\langle M^u, m\rangle_t = \int_0^t H_s(u_s, m)\,\mathrm{d}K_s, \quad t \in [0,T] \tag{41}$$

(for the existence proof of M^u see in Chitashvili and Mania, 1987a).

It follows immediately from the definition M^u, that the stochastic line integral does not depend on the choice of the dominating process K and for any $u \in U^0$, where U^0 is a set of controls taking values in some finite subset $\bar{A}$ of A,

$$M^u_t = \sum_{a \in \bar{A}} \int_0^t I_{(u_s = a)}\,\mathrm{d}M^a_s. \tag{42}$$

Without any loss of generality, we can assume that $P \in P^U$, i.e. $P = P^{u^0}$ for some $u^0 \in U$. The stochastic line integrals admit the properties

Proposition 3. *Let conditions* (C2)–(C4) *be satisfied. Then*

(1) *For any finite sequence* $(u^i,\ i \leqslant n) \in U$ *and any finite sequence* $(B_i,\ i \leqslant n)$ *of disjoint predictable sets there exists* $u \in U$ *such that*

$$M^u_t = \sum_{i=1}^n \int_0^t I_{B_i}(s)\,\mathrm{d}M^{u^i}_s.$$

In particular, the class M^U *is closed with respect to bifurcation.*

(2) *For any* $u, v \in U$

$$\langle M^u, M^v\rangle_t = \int_0^t \varphi(s, u_s, v_s)\,\mathrm{d}K_s. \tag{43}$$

(3) *If in addition condition* (C5) *is satisfied, then for every* $u \in U$

$$\Delta M^u_\tau = M(\tau, u(\tau)) - M(\tau-, u(\tau)) \tag{44}$$

for any stopping time τ.

Proof. (1) It follows immediately from the definition of line integrals, if we take $u_t = \sum_{i=1}^n u^i_t I_{B^i}(t) + u_0 I_{(\cup_{i=1}^n B_i)^c}$.

(2) Applying equality (41) for $m=M^v$ we have $\langle M^u,M^v\rangle_t=\int_0^t H(s,u_s,M^v)\,\mathrm{d}K_s$, where

$$H(s,u_s,M^v)=H(s,a,M^v)|_{a=u_s} \tag{45}$$

is a substitution of u in the function $H(a,M^v)$. On the other hand,

$$H(a,M^v)=\frac{\mathrm{d}\langle M^a,M^v\rangle}{\mathrm{d}K}=\frac{\mathrm{d}(H(v,M^a)\cdot K)}{\mathrm{d}K}=H(v,M^a),\quad \mu^K\text{-a.s.}$$

and $H(v,M^a)=H(b,M^a)|_{b=v}=\varphi(b,a)|_{b=v}=\varphi(v,a)$ (since $H(b,M^a)=\varphi(a,b)$). Therefore, from (45) we obtain that $H(u,M^v)=\varphi(u,v)$ and equality (43) holds. In particular, if $u=v$ we obtain

$$\langle M^u,M^u\rangle_t=\int_0^t \varphi_s(u_s)\,\mathrm{d}K_s. \tag{46}$$

(3) By localization we may assume that

$$E\int_0^T \sup_{a\in A}\varphi(s,a)\,\mathrm{d}K_s<\infty. \tag{47}$$

Since A is a compact there is a sequence $(u^i,\ i\geqslant 1)$ from U^0 such that $u^i\to u$.

Condition (C4) implies that $\psi(u^i,u)\to 0$ μ^K-a.e. Since by (43) and (39)

$$\langle M^{u^i}-M^u,M^{u^i}-M^u\rangle_t=\int_0^t \psi_s(u^i_s,u_s)\,\mathrm{d}K_s$$

and $\psi(u^i,u)=\varphi(u^i)-2\varphi(u^i,u)+\varphi(u)\leqslant 4\sup_{a\in A}\varphi(a)$ (μ^K-a.e.), by (47) and from the Lebesgue theorem of majorizing convergence we obtain that

$$E\langle M^{u_i}-M^u,M^{u_i}-M^u\rangle_T\to 0,\quad i\to\infty. \tag{48}$$

It follows immediately from (42) that for each $i\geqslant 1$

$$\Delta M^{u^i}_\tau=M(\tau,u^i(\tau))-M(\tau-,u^i_\tau)). \tag{49}$$

Applying the Doob inequality, from (48) we have that $E\sup_{s\leqslant T}(M^{u^i}_s-M^u_s)^2\to 0$, $i\to\infty$, hence,

$$\Delta M^{u^i}_\tau\to\Delta M^u_\tau,\quad i\to\infty, \tag{50}$$

in L^2 and P-a.s. for some subsequence of u_i.

On the other hand, the continuity of ΔM^a with respect to a and equality (49) implies that P-a.s. for each $n\geqslant 1$

$$\Delta M^{u^i}_\tau\to M(\tau,u(\tau))-M(\tau-,u(\tau)),\quad i\to\infty. \tag{51}$$

Therefore, from (50) and (51) we obtain the validity of equality (44). □

Thus, the controlled process is a family of measures $(P^u,\ u\in\bar U)$, where $\bar U=U\cap\{u\colon E\rho^u_T=1\}$, defined by $\mathrm{d}P^u=\mathscr{E}_T(M^u)\,\mathrm{d}P$, where M^u is the stochastic line integral. Under condition (C3*) $\bar U=U$ and the controlled measures P^u are well defined. The condition $P^a\sim P$ implies that $\Delta M^a_t>-1$ for every $a\in A$ and by property (44) of line integrals we have $\Delta M^u_t>-1$ for every $u\in U$ and $t\in[0,T]$. Therefore, the densities

$\rho_t^u = \mathscr{E}_t(M^u)$ are non-negative and under conditions (C1)–(C6) all controlled measures are equivalent to the measure P. Finally, suppose that

(C1) the random variable η is F_T-measurable and such that

$$\sup_{u \in U} E^u|\eta| < \infty, \quad E\eta^2 < \infty.$$

Theorem 3. *Let conditions* (C1)–(C6) *and* (C3*) *be satisfied. Then the value process* $S_t = \operatorname{esssup}_{u \in U} E^u(\eta/F_t)$, $t \in [0,T]$, *is a unique solution* (*in the class* $D(P^U)$) *of the equation*

$$\mathrm{d}Y_t = \left(\sup_{a \in A} H(t,a,m)\right) \mathrm{d}K_t + \mathrm{d}m_t, \quad m \in \mathscr{M}_{\mathrm{loc}} \tag{52}$$

with the boundary condition $Y_T = \eta$.

Besides, the optimal control u^* *exists and it may be constructed by pointwise* (*i.e., for any* t *and* ω) *maximization of the Hamiltonian* $H(a,m)$

$$H(t,u_t^*,m) = \sup_{a \in A} H(t,a,m). \tag{53}$$

Proof. We should show that the set of measures $\mathscr{Q} = P^U = (P^u,\ u \in U)$ satisfies conditions (B), (D*), (E), (F*) and ($A1$) of Theorem 2 and that μ^K-a.s.

$$\sup_{a \in A} \frac{\mathrm{d}\langle M^a, m\rangle}{\mathrm{d}K} = \operatorname*{esssup}_{u \in U} \frac{\mathrm{d}\langle M^u, m\rangle}{\mathrm{d}K}. \tag{54}$$

Condition (B) follows from Proposition 3 (property 1).

Condition (D*) (and $A1$) follows from conditions (C2), (C3*) and from Proposition 3 (property 2), since $\varphi(u) \leqslant \sup_{a \in A} \varphi(a)$.

Condition (E) follows from (C6) and Proposition 3 (property 3).

Let us show that condition (F*) is also satisfied. It is obvious that μ^K-a.e.

$$\sup_{a \in A} H(t,a,m) \geqslant \operatorname*{esssup}_{u \in U} H(t,u_t,m). \tag{55}$$

Since the functions $H(a,m)$ is μ^K-a.e. continuous in a and the decision set A is compact, by a measurable selection theorem (Benesh, 1971) a predictable function $u^* = (u_t,\ t \in [0,T]) \in U$ exists such that μ^K-a.e. equality (53) holds.

Thus, (53) and (55) imply that condition (F*) is satisfied (with $\varepsilon = 0$). Since, $\mathrm{d}\langle M^u, m\rangle/\mathrm{d}K = H(u,m)$ μ^K-a.e. by Definition of line integrals, (53) and (55) imply also that equality (54) holds. Therefore, Theorem 3 follows from Theorem 2 and the strategy u^* is optimal by Corollary 1. □

5. Determination of the maximum price of a contingent claim

Let $X = (X^1, X^2, \ldots, X^d)$ be a discounted price process of d assets, which is assumed to be a vector valued RCLL locally bounded process. A probability measure Q is called a local martingale measure, if it is equivalent to P and X is a Q-local martingale. Suppose that the set $\mathscr{P}(X)$ of local martingale measures is not empty, which corresponds to the absence of arbitrage opportunities on the security market (see Harison and Pliska, 1981; Delbaen and Schachermayer, 1994). For simplicity we assume that $P \in \mathscr{P}(X)$.

For each $Q\in\mathscr{P}(X)$ denote $Z^Q=(Z_t^Q=\mathrm{d}Q_t/\mathrm{d}P_t,\ t\geqslant 0)$ the density process, which is expressed as an exponential martingale

$$Z^Q=\mathscr{E}(M^Q)=(\mathscr{E}_t(M^Q),\ t\geqslant 0),\quad Q\in\mathscr{P}(X),$$

where $M^Q\in\mathscr{M}_{\mathrm{loc}}(F,P)$ and $\mathscr{E}(M)$ is the Dolean exponent.

Denote $\mathscr{M}(X)=(M^Q,Q\in\mathscr{P}(X))$, $Z(X)=(Z^Q,Q\in\mathscr{P}(X))$. Note that, since X is locally bounded, $\langle X,M\rangle$ exists for any $M\in\mathscr{M}_{\mathrm{loc}}$.

Each martingale M^Q associated with the martingale measure $Q\in\mathscr{P}(X)$ satisfies the following properties:

(1) $\Delta M_t^Q=M_t^Q-M_{t-}^Q>-1$, P-a.s., $t\in[0,T]$,

(2) $(\mathscr{E}_t(M^Q),\ t\in[0,T])$ is a martingale under P and

(3) $\langle M^Q,X^i\rangle=0$ for all $i\in\{1,2,\dots,d\}$,

where the last property follows from Girsanov's theorem.

Conversely, if M is some local martingale satisfying (1)–(3) then the measure Q equivalent to P, which admits $\mathscr{E}_T(M)$ as a Radon–Nicodým derivative relative to P on F_T, will be a local martingale measure of the process X.

Lemma 4. *The class of a local martingale measures is closed with respect to bifurcation, i.e. for $\mathscr{Q}=P(X)$ condition* (B) *is satisfied.*

Proof. For any $Q_1,Q_2\in\mathscr{P}(X)$, a stopping time τ and $B\in F_\tau$ define the measure Q by $\mathrm{d}Q=\mathscr{E}_T(M)\,\mathrm{d}P$, where $M_t=0$ for $t\leqslant\tau$ and

$$M_t=(M_t^{Q_1}-M_\tau^{Q_1})I_B+(M_t^{Q_2}-M_\tau^{Q_2})I_{B^c} \tag{56}$$

on the set $t>\tau$. It is easy to see that the martingale M satisfies conditions (1)–(3) given above. It follows from (56) that $\mathscr{E}_t(M)=1$ for $t\leqslant\tau$ and

$$\mathscr{E}_t(M)=\mathscr{E}_t(M^{Q_1})\mathscr{E}_\tau^{-1}(M^{Q_1})I_B+\mathscr{E}_t(M^{Q_2})\mathscr{E}_\tau^{-1}(M^{Q_2})I_{B^c}$$

if $t\geqslant\tau$. Therefore, it is easily verified that $\mathscr{E}(M)$ is a P-martingale. It follows from (56) that $\langle M,X\rangle_t=0$, $\Delta M>-1$ and, hence Q is a local martingale measure. □

Thus, according to Proposition 1 there exists an RCLL modification of the maximum price process V and it is the smallest supermartingale relative to any $Q\in\mathscr{P}(X)$, which is equal to η at time T. Besides, the martingale measure Q^* is optimal iff V is a martingale under Q^*. Moreover, it was proved by Ansel and Stricker (1994) that the optimal martingale measure Q^* exists if and only if η is Q^* attainable, i.e. iff one can hedge the contingent claim η by a self-financing portfolio. In particular, if an optimal martingale measure Q^* exists then the value process admits an integral representation $V_t=E^{Q^*}(\eta/F_t)=E^{Q^*}\eta+\int_0^t H_s\,\mathrm{d}X_s$, where $\int_0^t H_s\,\mathrm{d}X_s$ is a stochastic integral of predictable process H with respect to the vector valued semimartingale X (see Jacod (1979) for precise definitions). If the market is incomplete, it is not possible to replicate all contingent claims by a controlled portfolio of the basic securities. But, as shown by Kramkov (1996), the hedging strategy with consumption under the present assumptions always exists. Namely, it was proved in Kramkov (1996) that if (A) is satisfied and the process X is locally bounded then the hedging strategy with consumption exists and

$$V_t = V_0 + \int_0^t H_s \, \mathrm{d}X_s - C_t, \quad t \in [0,T], \tag{57}$$

where $H = (H^i,\ i \leqslant d)$ is a predictable X-integrable process of numbers of assets and $C = (C_t,\ t \in [0,T])$ is an adapted increasing process of consumption. This theorem (called an optional decomposition of the wealth process) was first proved by El Karoui and Quenez (1995) in the case of diffusion market model. The proof of this result for general semimartingales is given in a recent paper of Föllmer and Kabanov (1998).

The optional decomposition (57) is invariant relative to $Q \in \mathscr{P}(X)$ and it gives a representation of the value process as a controlled portfolio with the consumption. But for the calculation of H and C in (57) a derivation of a differential equation for the value process is desirable, for which the predictable decomposition of V should be used.

We assume:

(F1) A contingent claim η is a positive, F_T-measurable random variable satisfying

$$\sup_{Q \in \mathscr{P}(X)} E^Q \eta < \infty \quad \text{and} \quad E\eta^2 < \infty.$$

(F2) $M^Q \in \mathscr{M}^2_{\mathrm{loc}}(F,P)$ for each $Q \in \mathscr{P}(X)$ and there exists a predictable bounded increasing process $K = (K_t,\ t \in [0,T])$ such that $\langle M^Q, M^Q \rangle \ll K$, $Q \in \mathscr{Q}$.

(F3) For any $Q \in \mathscr{Q}$ there is a sequence $(\tau_n,\ n \geqslant 1)$ of stopping times (which may depend on Q) such that $\cup_n(\tau_n = T) = \Omega$ a.s. and

$$\langle M^Q \rangle_{\tau_n} \leqslant n, \quad \inf_{t \leqslant \tau_n} \Delta M_t^Q \geqslant \frac{1}{n} - 1$$

for each $n \geqslant 1$.

Remark 1. Condition (F3) is satisfied if, e.g. $(M_t^Q,\ t \in [0,T])$ is continuous, since $Q_T \sim P_T$ implies that $\langle M^Q \rangle_T < \infty$ a.s.

Denote by V_t the maximum of the possible prices of a contingent claim η at time t

$$V_t = \operatorname*{esssup}_{Q \in \mathscr{P}(X)} E^Q(\eta / F_t) = \operatorname*{esssup}_{M \in \mathscr{M}(X)} \mathscr{E}_t^{-1}(M) E(\eta \mathscr{E}_T(M)/F_t).$$

Let

$$\mathscr{M}^n(X) = \left\{ M^Q \in \mathscr{M}(X)\text{: } \frac{\mathrm{d}\langle M^Q \rangle}{\mathrm{d}K} \leqslant n, \Delta M^Q \geqslant \frac{1}{n} - 1 \right\},$$

$$Z^n(X) = \{\mathscr{E}(M^Q)\text{: } M^Q \in \mathscr{M}^n(X)\}$$

and let

$$\mathscr{P}^n(X) = \{Q \in \mathscr{P}(X)\text{: } \mathrm{d}Q = \mathscr{E}_T(M^Q), M^Q \in \mathscr{M}^n(X)\}.$$

Denote by V_t^n a right continuous process satisfying

$$V_t^n = \operatorname*{esssup}_{Q \in \mathscr{P}^n(X)} E^Q(\eta / F_t) \quad \text{a.s.} \tag{58}$$

The class $\mathscr{P}^n(X)$ of martingale measures is stable with respect to bifurcation and, therefore, the process V_t^n is also characterized by Proposition 1 as the smallest right continuous supermartingale with respect to any $Q\in\mathscr{P}^n(X)$ with $V_T^n=\eta$.

Theorem 4. *Let conditions* (F1)–(F3) *be satisfied. Then* $\lim_{n\to\infty} V_t^n = V_t$ *a.s. for each* $t\in[0,T]$ *and the process* V^n *is the unique solution* (*in the class* $D(\mathscr{P}^n(X))$) *of the backward semimartingale equation*

$$\mathrm{d}Y_t^n = -\operatorname*{esssup}_{Q\in\mathscr{P}^n(X)} \frac{\mathrm{d}\langle Y^n, M^Q\rangle_t}{\mathrm{d}K_t}\,\mathrm{d}K_t + \mathrm{d}m_t^n, \quad m^n\in\mathscr{M}_{\mathrm{loc}}(P) \tag{59}$$

with the boundary condition $Y_T=\eta$.

Proof. Let $\tilde V_t = \lim_{n\to\infty} V_t^n$. It is obvious that $\tilde V_t\leqslant V_t$. Therefore, to show that $\lim_{n\to\infty} V_t^n = V_t$ a.s. for each $t\in[0,T]$, it is sufficient to prove that $\tilde V_t \geqslant V_t$ a.s. $\tilde V_t$ is an RCLL supermartingale relative to all $Q\in\mathscr{P}^n(X)$ and all $n\geqslant 1$, since it is an increasing limit of RCLL supermartingales, hence,

$$\tilde V_t \geqslant E^Q(\eta/F_t) \quad \text{a.s.} \tag{60}$$

for all $Q\in\mathscr{P}^n(X)$ and all $n\geqslant1$. Let us first show that inequality (60) holds for any $Q\in\mathscr{P}(X)$ with $\langle M^Q\rangle_T\leqslant C$ and $\Delta M^Q\geqslant 1/n-1$.

Let $M_t^{Q_n}=\int_0^t I_{(\mathrm{d}\langle M^Q\rangle/\mathrm{d}K\leqslant n)}(s)\,\mathrm{d}M_s^Q$. It is evident that $\mathrm{d}\langle M^{Q_n}\rangle/\mathrm{d}K\leqslant n$ and the measure Q_n with $\mathrm{d}Q_n = \mathscr{E}(I_{(\mathrm{d}\langle M^Q\rangle/\mathrm{d}K\leqslant n)}M^Q)\,\mathrm{d}P$ will be a martingale measure from the class $\mathscr{P}^n(X)$. It is evident that $\langle M^{Q_n}\rangle_T\leqslant\langle M^Q\rangle_T\leqslant C$ for all n and

$$P(\langle M^{Q_n}-M^Q\rangle_T\geqslant\varepsilon)\to 0, \quad n\to\infty. \tag{61}$$

Since (61) implies (A.7), it follows from Proposition A.2 that

$$E^{Q_n}(\eta/F_t)\to E^Q(\eta/F_t), \quad n\to\infty, \tag{62}$$

in probability and using (62) by the passage to the limit in inequality (60) for Q^n we obtain that inequality (60) is satisfied for any $Q\in\mathscr{P}(X)$ with $\langle M^Q\rangle_T\leqslant C$ and $\Delta M^Q\geqslant c>-1$.

Now let Q be an arbitrary element of $\mathscr{P}(X)$ and let $(\tau_n, n\geqslant1)$ be a sequence of stopping times from condition (F3).

Let $M_t^n = M_{t\wedge\tau_n}^Q$ and $\mathrm{d}Q^n=\mathscr{E}_T(M^n)\,\mathrm{d}P$. Since $\langle M^n\rangle_T\leqslant n$ and $\Delta M^n\geqslant 1/n-1$ we have that

$$\tilde V_t\geqslant E^{Q^n}(\eta/F_t) = \mathscr{E}_{t\wedge\tau_n}^{-1}(M^Q)E(\eta\mathscr{E}_{T\wedge\tau_n}(M^Q)/F_t). \tag{63}$$

Now using Fatou's Lemma, condition (F3), the martingale convergence theorem and relation $\cup_n(\tau_n=T)=\Omega$ a.s., by passage to the limit in (63) we obtain that inequality (60) holds for any martingale measure Q, hence

$$\tilde V_t\geqslant \operatorname*{esssup}_{Q\in\mathscr{Q}} E^Q(\eta/F_t) = V_t.$$

Let us show that the family of measures $\mathscr{P}^n(X)$ satisfies condition (B*).

For any sequence $(Q_i,\ i\geqslant1)\in\mathscr{P}^n(X)$ and any sequence of predictable sets $(B_i,\ i\geqslant1)$, with $B_i\cap B_j=\emptyset$ for $i\neq j$ and $\cup_i B_i=[0,T]\times\Omega$, let us consider a sequence of martingales $M_t^j=\sum_{i=1}^j I_{B^i}\cdot M^{Q_i}$. Evidently, $M^j\in\mathscr{M}^n(X)$ for every $j\geqslant 1$ and $(M^j,j\geqslant 1)$ is the Cauchy sequence in L^2. Indeed,

$$E(M_T^j-M_T^k)^2=E\sum_{i=k}^{j}\int_0^T I_{B^i}(t)\,\mathrm{d}\langle M^{Q_i},M^{Q_i}\rangle_s\leqslant nE\int_0^T I_{(\cup_{i=k}^j B_i)}(t)\,\mathrm{d}K_t\to 0$$

as $k,j\to\infty$. Therefore, there exists $M\in\mathscr{M}^2$ such that $\lim_{j\to\infty}E\langle M^j-M\rangle_T=0$ and Lemma A.2 implies that $\mathrm{d}\langle M\rangle/\mathrm{d}K\leqslant n\ \mu^K$-a.e. On the other hand $\Delta M_t^j\geqslant 1/n-1$ and the Doob inequality implies that $\lim_{j\to\infty}E\sup_{t\in[0,T]}|M_t^j-M_t|=0$, hence $\Delta M\geqslant 1/n-1$.

Since $\langle M^j,X\rangle=0$ for every $j\geqslant 1$ by passage to the limit we have that $\langle M,X\rangle=0$. Thus, $M\in\mathscr{M}^n(X)$ and passing to the limit (by j) in the equality

$$\int_0^T I_{B_i}(s)\,\mathrm{d}M^j=\int_0^T I_{B_i}(s)\,\mathrm{d}M_s^i$$

which is valid for every $j\geqslant i$, we obtain that the martingale M satisfies (30). Evidently, $(\mathscr{E}_t(M),t\in[0,T])$ is a P-martingale and the measure Q defined by $\mathrm{d}Q=\mathscr{E}_T(M)\,\mathrm{d}P$ belongs to $\mathscr{P}^n(X)$. Thus the set of measures $\mathscr{P}^n(X)$ satisfies condition (B*) and by Lemma 2 condition (F*) is fulfilled. It is evident that the other conditions of Theorem 2 are also satisfied. Hence, the proof of theorem follows from Theorem 2. □

In fact, the class of densities $Z^n(X)$ is weakly compact in L^2 and the supremum in $\sup_{Q\in\mathscr{P}^n(X)}E^Q\eta$ is attained for every $n\geqslant 1$.

Proposition 4. *The class of densities $Z^n(X)$ is weakly compact in L^2.*

Proof. Evidently, $Z^n(X)$ is strongly bounded subset of L^2 and it follows from Lemma A.1 that $Z^n(X)$ is convex.

Let us show that $Z^n(X)$ is strongly closed. Let the sequence $(Q^i,\ i\geqslant 1)\in\mathscr{P}^n(X)$ be such that $E(\mathscr{E}_T(M^{Q_i})-Z)^2\to 0$, where Z is some element of $\mathscr{M}^2$. Since $\mathscr{E}_T(M^{Q_i})$ is the Cauchy sequence in L^2, inequalities $\mathrm{d}\langle M^{Q_i}\rangle/\mathrm{d}K\leqslant n$, $i\geqslant 1$, and Proposition A.1 implies that $M^{Q_i}\in\mathscr{M}^n(X)$ will be a Cauchy sequence in $\mathscr{M}^n(X)$. By Lemma A.2 the class $\mathscr{M}^n(X)$ is strongly closed and $E\langle M^{Q_i}-M\rangle_T\to 0$, for some $M\in\mathscr{M}^n(X)$. Using the sufficiency part of Proposition A.1 we have $Z=\mathscr{E}(M)$ which implies that $Z\in Z^n(X)$, hence $Z^n(X)$ is weakly compact. □

Remark 1. Theorem 4 enables us to construct the ε-optimal martingale measures. For some $\varepsilon>0$ one can find $n=n(\varepsilon)$ for which $V_0^n=\sup_{Q\in\mathscr{Q}^n}E^Q\eta\geqslant\sup_{Q\in\mathscr{Q}}E^Q\eta-\varepsilon$ and then we construct $Q^{n,*}$ so that $E^{Q^{n,*}}\eta=\sup_{Q\in\mathscr{Q}^n}E^Q\eta$. The martingale measure $Q^{n,*}$ may be constructed by the maximization of the expression $H(Q,V^n)$.

Remark 2. It is easy to see that if an optimal martingale measure exists then it follows from Theorem 4 and from the theorem of Ansel and Stricker (1994) that $A_t^n\to^P 0$, where $A_t^n=\int_0^t \operatorname{esssup}_{Q\in\mathscr{Q}^n}(\mathrm{d}\langle M^Q,V\rangle_s/\mathrm{d}K_s)\,\mathrm{d}K_s$.

Let W be a d-dimensional Brownian motion under P and assume that $F=(F_t^W, t\in[0,T])$ is filtration generated by W. Suppose that the stock price process X is governed by the SDE

$$\mathrm{d}X_t^i = X_t^i \left(\sum_{i=1}^{d} \sigma_{i,j}(t)\, \mathrm{d}W_t^j \right). \tag{64}$$

The volatility matrix σ is assumed to be bounded and F^W-predictable (for simplicity we assume that the interest and appreciation rates are equal to zero).

Using the notations of El Karoui and Quenez (1995) we have in this case, that

$$\mathscr{P}^n(X) = \left\{ Q_v\colon\ \mathrm{d}Q_v = \mathscr{E}_T \left(\int_0^{\cdot} v_s\, \mathrm{d}W_s \right),\ v \in \mathscr{K}^n(\sigma) \right\},$$

where $\mathscr{K}^n(\sigma)$ is a class of predictable processes v such that $||v|| \leqslant n$ and $v\sigma = 0$ $\mathrm{d}P \times \mathrm{d}t$-a.e. Let ψ^n be an integrand of the martingale part in the decomposition of $V_t^n = \operatorname{esssup}_{v \in \mathscr{K}^n(\sigma)} E^{Q_v}(\eta/F_t)$ under the measure P.

The following statement was proved in El Karoui and Quenez (1995) using the results of existence and uniqueness of backward equations of Pardoux and Peng (1990), where the solution is constructed by a Picard-type iteration.

Corollary. *Let condition* (F1) *be satisfied. Then* $\lim_{n\to\infty} V_t^n = V_t$ *a.s. for each* $t \in [0, T]$ *and* (V_t^n, ψ_t^n) *is the unique solution of the backward SDE*

$$Y_t^n - \int_t^T n||\pi_{\mathrm{Ker}\,\sigma_s}(g_s^n)||\, \mathrm{d}s + \int_t^T g_s^n\, \mathrm{d}W_s = \eta, \quad t \in [0, T], \tag{65}$$

where $\pi_{\mathrm{Ker}\,\sigma_s}$ *denotes the orthogonal projection that maps* $\mathbb{R}^d$ *onto the kernel of* σ_s.

Proof. It is evident that all conditions of Theorem 4 are satisfied and it is easy to see that in this case Eq. (36) is equivalent to (65), since

$$\operatorname*{esssup}_{Q \in \mathscr{P}^n(X)} \frac{\mathrm{d}\langle M^Q, m\rangle_s}{\mathrm{d}K_s} = \operatorname*{esssup}_{v \in \mathscr{K}^n(\sigma)} v_s \psi_s^n = n||\pi_{\mathrm{Ker}\,\sigma_s}(\psi_s^n)||.$$

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Appendix

For convenience, we give here some simple lemmas and known propositions used in this paper.

Lemma A.1. *Let H and K be predictable processes such that $K \in \mathscr{A}^+_{\mathrm{loc}}$, $H \geqslant 0$ and $\int_0^T H(s)\,\mathrm{d}K_s \leqslant C$ (a.s.) for some constant $C < \infty$. Let $(M^i, i \leqslant n) \in \mathscr{M}^2_{\mathrm{loc}}$ and for each $i \leqslant n$*

$$M_0^i = 0, \quad \Delta M^i > -1, \quad \langle M^i \rangle \ll K \quad \textit{and} \quad \mathrm{d}\langle M^i \rangle / \mathrm{d}K \leqslant H \quad \mu^K\text{-a.e.} \tag{A.1}$$

For any $\lambda = (\lambda_1, \dots, \lambda_n)$ with $\lambda_i \geqslant 0$ for all i and $\sum_i^n \lambda_i = 1$ there exists a local martingale M satisfying (A.1) *such that $\mathscr{E}_t(M) = \sum_i^n \lambda_i \mathscr{E}_t(M^i)$.*

Proof. It is obvious that $Z_t = \sum_i^n \lambda_i \mathscr{E}_t(M^i)$ satisfies Eq. (1) with

$$M_t = \sum_i^n \int_0^t (\lambda_i \mathscr{E}_{s-}(M^i)/Z_{s-})\,\mathrm{d}M_s^i. \tag{A.2}$$

Evidently, $\Delta M > -1$ and using arguments similar to the Remark 2 of Section 3 it is easy to see that $\mathrm{d}\langle M \rangle / \mathrm{d}K \leqslant H$ (μ^K)-a.e. □

Remark. It is obvious that if $\langle M^i, X \rangle = 0$, $i \leqslant n$, for some $X \in M^2_{\mathrm{loc}}$ then $\langle M, X \rangle = 0$.

The proof of the following lemma is also obvious.

Lemma A.2. *Let H and K be the same as in Lemma* A.1. *Let $(M^i, i \geqslant 1)$ be a sequence of locally square integrable martingales and for every $\varepsilon > 0$*

$$P(\langle M^i - M \rangle_T \geqslant \varepsilon) \to 0, \quad i \to \infty \tag{A.3}$$

for some $M \in \mathscr{M}^2_{\mathrm{loc}}$. Then

(a) *if $\langle M^i \rangle \leqslant C$ (a.s.) for every $i \geqslant 1$ then $\langle M \rangle \leqslant C$ (a.s.),*
(b) *if $\langle M^i \rangle \ll K$ for every $i \geqslant 1$ then $\langle M \rangle \ll K$,*
(c) *if $\mathrm{d}\langle M^i \rangle / \mathrm{d}K \leqslant H$ (μ^K-a.e.) for every $i \geqslant 1$ then $\mathrm{d}\langle M \rangle / \mathrm{d}K \leqslant H$ (μ^K-a.e.),*
(d) *if $\langle M^i, X \rangle = 0$ for every $i \geqslant 1$ then $\langle M, X \rangle = 0$, for any $X \in \mathscr{M}^2_{\mathrm{loc}}$.*

Proposition A.1. *Let M, $(M^i, i \geqslant 1) \in \mathscr{M}^2_{\mathrm{loc}}$ and*

$$\langle M^i, M^i \rangle_T \leqslant C \quad \textit{for every } i \geqslant 1. \tag{A.4}$$

Then

$$E(\mathscr{E}_T(M^i) - \mathscr{E}_T(M))^2 \to 0, \quad i \to \infty, \tag{A.5}$$

if and only if

$$P(\langle M^i - M \rangle_t \geqslant \varepsilon) \to 0, \quad n \to \infty. \tag{A.6}$$

The proof of Proposition 1 can be seen in Chitashvili and Mania (1987b). Since under condition (A.4) convergence (A.5) is equivalent to the convergence $\mathscr{E}(M^i) \to \mathscr{E}(M)$ in L^1, one can deduce this assertion from Kabanov et al. (1986) also.

Proposition A.2 (Chitashvili and Mania, 1987b). *Let M, $(M^i,\ i \geqslant 1) \in \mathscr{M}^2_{\mathrm{loc}}$,* (A.4) *is satisfied and $E\eta^2 < \infty$. If*

$$P(|\langle M^i - M, m \rangle_T| \geqslant \varepsilon), \quad n \to \infty, \tag{A.7}$$

for any $m \in \mathscr{M}^2$, *then*

$$\mathscr{E}_\tau^{-1}(M^i)E(\mathscr{E}_T(M^i)\eta/F_\tau) \to \mathscr{E}_\tau^{-1}(M)E(\mathscr{E}_T(M)\eta/F_\tau), \quad i \to \infty \tag{A.8}$$

in probability, *for any stopping time* τ.

References

Ansel, J.P., Stricker, C., 1994. Couverture de actifs contingents et prix maximum. Ann. Inst. Poincare 30, 303–315.

Benesh, V.E., 1971. Existence of optimal stochastic controll laws. SIAM J. Control Optim. 9, 446–472.

Chitashvili, R., 1983. Martingale ideology in the Theory of Controlled Stochastic Processes. Lecture Notes in Mathematics, Vol. 1021, Springer, Berlin, pp. 73–92.

Chitashvili, R., Mania, M., 1987a. On optimal controls in the problem of locally absolutely continuous change of measure. Proceedings of the fourth International Vilnius Conference, Vol. 1, VNU Science Press, Vilnius, pp. 331–356.

Chitashvili, R., Mania, M., 1987b. Optimal locally absolutely continuous change of measure: finite set of decisions. Stochastics, 21 (part 1), 131–185, (part 2), 187–229.

Chitashvili, R., Mania, M., 1996. Generalized Ito's formula and derivation of Bellman's equation. In: Stochastic Processes and Related Topics, Stochastic Monographs, Vol. 10, (Eds. H.J. Engelbert, H. Föllmer, J. Zabczyk) Gordon & Breach Science Publishers, London, pp. 1–21.

Davis, M.H.A., Varaiya, P., 1973. Dynamic programming conditions for partially observable stochastic systems. SIAM J. Control Optim. 2, 226–261.

Delbaen, F., Schachermayer, W., 1994. A general version of the fundamental theorem of asset pricing. Math. Annalen 300, 463–520.

Dellacherie, C., Meyer, P.A., 1980. Probabilités et potentiel II. Hermann, Paris.

Elliott, R.J., 1982. Stochastic Calculus and Applications. Springer, Berlin.

Harison, J.M., Pliska, S.R., 1981. Martingales and stochastic integrals in the theory of continuous trading. Stochastic Process. Appl. 11, 215–260.

Jacod, J., 1979. Calcule Stochastique et problèmes des martingales. Lecture Notes in Mathematics, Vol. 714. Springer, Berlin.

El Karoui, N., Quenez, M.C., 1995. Dynamic programming and pricing of contingent claims in an incomplete market. SIAM J. Control Optim. 33 (1), 29–66.

Kabanov, Yu., Liptser, R.S., Shiryaev, A.N., 1986. On the variation distance for probability measures defined on a filtered space. Probab. Theory Related Fields 71, 19–36.

Kramkov, D.O., 1996. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. Probab. Theory Related Fields 105, 459–479.

Liptzer, R.Sh., Shiryayev, A.N., 1986. Martingale Theory. Nauka, Moscow.

Pardoux, E., Peng, S.G., 1990. Adapted solution of a backward stochastic differential equation. Systems Control Lett. 14, 55–61.

Föllmer, H., Kabanov, Ju.M., 1998. Optional decomposition and Lagrange multipliers. Finance Stochastics 2 (1), 69–81.